Properties of Bel currents

R. Lazkoz

Facultad de Ingeniería (ESIDE), Universidad de Deusto, Apdo. de Correos, 1, 48080 Bilbao, Spain

J.M.M. Senovilla

Fisika Teorikoaren Saila, Euskal Herriko Unibertsitatea, 644 P.K., 48080 Bilbao, Spain

R. Vera

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, England, UK

The Bel tensor is divergence-free in some important cases leading to the existence of conserved currents associated to Killing vectors analogously to those of the energy-momentum tensor. When the divergence of the Bel tensor does not vanish one can study the interchange of some quantities between the gravitational and other fields obtaining mixed total conserved currents. Nevertheless, the Bel currents are shown to be conserved (independently of the matter content) if the Killing vectors satisfy some very general conditions. These properties are similar to some very well known statements for the energy-momentum tensor.

1 Introduction

A consequence of the Principle of Equivalence is that there is no possible proper definition of a *local energy-momentum tensor* for the gravitational field, where by "proper" it is meant a tensor constructed from the metric and its first derivatives. Nevertheless, there exist *local* tensors describing the *strengh* of the gravitational field. The outstanding example is the so called Bel–Robinson (B-R) tensor [1,2], a four-index tensor constructed for vacuum spacetimes whose properties are similar to the traditional energy-momentum (e-m) tensors: it has analogous positivity properties, it vanishes iff the curvature does, it is divergence free, and others. But it is not an energy-momentum tensor: it has four indices, and its physical dimensions are not those of an energy density (it was Bel himself who refer to that dimensions as 'super-energy', whose best interpretation so far is that of energy per surface unit). Therefore, instead of trying to define a sort of 'energy' for the gravitational field, the idea behind the works on super-energy (s-e) tensors is to define analogous objects for the rest of the physical fields which ought to be related somehow at this s-e level. A purely algebraic construction of s-e tensors for arbitrary fields was presented in [3], and includes the usual Bel tensor, which generalizes the B-R tensor for non-vacuum spacetimes. We refer to [4] for a discussion on this issue. Contracting the Bel tensor with Killing vectors one constructs some 'Bel currents' which are not divergence-free in general (the matter acts as source), so that they may lead to the interchange of "s-e" quantities between the gravitational and other physical fields. This possibility is analyzed in [3] for the cases of the minimally coupling with a scalar field, electromagnetic and Proca fields, using as inspiration the <u>mixed</u> divergence-free currents (which are not conserved separately) traditionally found by adding the e-m tensors describing fields in interaction.

This s-e interchange has been found in spacetimes admitting a non orthogonally transitive G_2 group of isometries, in the Wils' family of stiff fluid solutions [5], see [6]. In simpler cases the Bel currents are shown [6] to be conserved automatically depending on some geometrical properties of the Killing vectors used in their construction, independently of the matter content [6]. These properties are the analogous to some very well known statements concerning the e-m tensors.

The covariant derivative associated to g in our spacetime $(\mathcal{V}, \mathrm{g})$ will be denoted both by ∇ and ;. Round and square brakets embracing any number of indices will denote the usual symmetrization and antisymmetrization, respectively. Greek indices run from 0 to 3.

2 The Bel tensor and its currents

The Bel tensor [1,2], which can be thought as being the basic super-energy tensor (T) for the gravitational field constructed with the Riemann tensor $R_{\alpha\beta\lambda\mu}$ (double symmetric (2,2)-form, i.e. $R_{[2],[2]}$) in four dimensions [3], reads as follows

$$B_{\alpha\beta\lambda\mu} \equiv T_{\alpha\beta\lambda\mu}\left\{R_{[2],[2]}\right\} = R_{\alpha\rho\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} -$$
$$-\frac{1}{2}g_{\alpha\beta}R_{\rho\tau\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma} - \frac{1}{2}g_{\lambda\mu}R_{\alpha\rho,\sigma\tau}R_{\beta}{}^{\rho\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau\sigma\nu}R^{\rho\tau\sigma\nu},$$

from where the following symmetry properties for B explicitly arise

$$B_{\alpha\beta\lambda\mu} = B_{(\alpha\beta)(\lambda\mu)} = B_{\lambda\mu\alpha\beta}. \tag{1}$$

Using the second Bianchi identity $\nabla_{[\nu}R_{\alpha\beta]\lambda\mu} = 0$ one obtains the following expression for the divergence of the Bel tensor

$$\nabla_\alpha B^{\alpha\beta\lambda\mu} = R^{\beta\ \lambda}_{\ \rho\ \sigma} J^{\mu\sigma\rho} + R^{\beta\ \mu}_{\ \rho\ \sigma} J^{\lambda\sigma\rho} - \frac{1}{2} g^{\lambda\mu} R^{\beta}_{\ \rho\sigma\gamma} J^{\sigma\gamma\rho}, \tag{2}$$

where $J_{\lambda\mu\beta} = -J_{\mu\lambda\beta} \equiv \nabla_\lambda R_{\mu\beta} - \nabla_\mu R_{\lambda\beta}$. Notice that because of (1) this is the only independent divergence of the Bel tensor. The fundamental result we have from (2) is that B is divergence-free when the 'current' source of matter $J_{\lambda\mu\beta}$ vanishes. This includes all Einstein spaces (where $R_{\mu\nu} = \Lambda g_{\mu\nu}$), so that in particular this implies that the Bel-Robinson tensor, which is just the specialization of the Bel tensor for vacuum, is divergence-free. The divergence-free property of B in these cases allows us to obtain conserved currents in the same way as it is usally done using the energy momentum tensor, once there exist Killing vector fields in our spacetime, as we will presently see.

These conserved currents are nothing but divergence-free vector fields, so that conserved quantities can be obtained by means of the Gauss theorem integrating over appropiate domains of the manifold $\mathcal{V}$ [7]. Following [3] one can define the current related to the Bel tensor with respect to three arbitrary Killing vector fields $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ as

$$j_\mu \left(R_{[2],[2]}; \vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3 \right) \equiv B_{(\alpha\beta\lambda)\mu}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda = B_{(\alpha\beta\lambda\mu)}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda. \tag{3}$$

The divergence of this current can be computed to give

$$\begin{aligned} \nabla_\mu j^\mu \left(R_{[2],[2]}; \vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3 \right) &= \nabla^\mu B_{(\alpha\beta\lambda\mu)}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda + 3 B_{(\alpha\beta\lambda\mu)} \nabla^{(\mu} \xi_{(1}^{\alpha)} \xi_2^\beta \xi_{3)}^\lambda \\ &= \nabla^\mu B_{(\alpha\beta\lambda\mu)}\, \xi_1^\alpha \xi_2^\beta \xi_3^\lambda, \end{aligned}$$

using the fact that $\vec{\xi}_{A\,(\alpha;\beta)} = 0$ $(A = 1, 2, 3)$ Therefore, the vanishing of the divergence of the Bel tensor implies the vanishing of that of $\vec{j} \left(R_{[2],[2]}; \vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3 \right)$, which constitutes then a conserved current.

But the divergence of the Bel tensor does not vanish in general. As mentioned in the Introduction, these currents, if not conserved, lead to the description of the interchange of some quantities between different physical systems, because it is the *total* quantity defined for the whole system which is indeed conserved [3]. The interchange of these quantities, constructed from the currents for the so-called super-energy tensors (s-e quantites, then) for the case of the coupling between the scalar and gravitational fields has been already shown in [3], see also [4], and explicit examples are given in [6], were neither $\vec{j}$ nor the current associated with the s-e tensor of a scalar field ϕ, $\vec{j}_\phi$, are conserved, but the mixed current $\vec{j} + \vec{j}_\phi$ is indeed divergence-free.

These examples of interchange of s-e, though, are to be found in spacetimes admitting a G_2 group of isometries acting on spacelike surfaces but not orthogonally transitively, the reason being that otherwise the s-e currents are independently conserved. This is due to the fact that the currents obtained from the s-e tensor for the gravitational field (Bel tensor) are conserved automatically depending on some geometrical properties of the Killing vectors used in its construction, independently of the form of the Ricci tensor and thus of the matter content. This property, far from being undesired, is what one would expect from a good generalization of the energy-momentum tensor Indeed, the properties we present later in section 4 that lead to the vanishing of the divergenge of $\vec{j}\left(R_{[2],[2]};\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3\right)$ are the analogous to some very well known statements involving both the Ricci tensor, and hence the energy-momentum tensor, and two (or one) Killing vector fields generating a G_2 or G_1 group of isometries (see [8,9,10,11,12,13]). Let us recall them altogether in the following theorem 3.1, just after giving some remarks.

3 Geometric properties of the energy-momentum currents

We say that a vetor field $\vec{v}$ is 'integrable' (or hypersurface orthogonal) when $\boldsymbol{v}\wedge \mathrm{d}\boldsymbol{v}=0$ (in components $v_{[\alpha}v_{\beta;\gamma]}=0$), and second, two non-null vector fields orthogonal to two given vectors $\vec{v}$ and $\vec{w}$ generate surfaces whenever the two 1-forms $\boldsymbol{v}$ and $\boldsymbol{w}$ associated to the vector fields $\vec{v}$ and $\vec{w}$ satisfy $\boldsymbol{v}\wedge\boldsymbol{w}\wedge \mathrm{d}\boldsymbol{w}=\boldsymbol{v}\wedge\boldsymbol{w}\wedge \mathrm{d}\boldsymbol{v}=0$.

When $\vec{\xi}$ and $\vec{\eta}$ are two non-null Killing vector fields, the group G_2 generated by them is said to act orthogonally transitively when the vector fields orthogonal to the group orbits generate surfaces. This means that $\boldsymbol{\xi}\wedge\boldsymbol{\eta}\wedge \mathrm{d}\boldsymbol{\eta}=\boldsymbol{\xi}\wedge\boldsymbol{\eta}\wedge \mathrm{d}\boldsymbol{\xi}=0$, which in components read $\xi_{[\alpha}\eta_\beta\eta_{\lambda;\rho]}=0, \eta_{[\alpha}\xi_\beta\xi_{\lambda;\rho]}=0$.

Defining the currents asociated with the usual energy-momentum tensor $T_{\alpha\beta}$, which equates the Einstein tensor $S_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}R\ \mathrm{g}_{\alpha\beta}$ via the Einstein field equations, with respect to a given Killing vector field $\vec{\xi}$, $\vec{\mathcal{J}}\left(\vec{\xi}\right)$, as $\mathcal{J}_\alpha\left(\vec{\xi}\right)\equiv\xi^\beta T_{\alpha\beta}$, one can present the known results above mentioned as follows:

Theorem 3.1 [8,9,10,11,12,13] *Let $\vec{\xi}$ be a non-null Killing vector field and $\vec{\mathcal{J}}\left(\vec{\xi}\right)$ its energy-momentum tensor current. If $\vec{\xi}$ is integrable, then $\mathcal{J}_{[\beta}\left(\vec{\xi}\right)\xi_{\lambda]}=0$.*

If the spacetime admits two independent non-null Killing vector fields $\vec{\xi}$ and $\vec{\eta}$, let $\vec{\mathcal{J}}\left(\vec{\xi}\right)$ and $\vec{\mathcal{J}}\left(\vec{\eta}\right)$ be their respective energy-momentum tensor currents.

If $[\vec{\xi},\vec{\eta}]=0$ *and the group acts orthogonally transitively, then* $\mathcal{J}_{[\beta}\left(\vec{\xi}\right)\xi_{\lambda}\eta_{\mu]}=\mathcal{J}_{[\beta}\left(\vec{\eta}\right)\xi_{\lambda}\eta_{\mu]}=0.$

4 Geometrical properties of the Bel currents

Following (3), when our spacetime admits two independent Killing vector fields $\vec{\xi}$ and $\vec{\eta}$, there appear 4 different Bel currents associated with them. Labeling our two Killing vectors as $\vec{\xi_L}$ (where $L, M, N = 1, 2$) such that $\vec{\xi_1}\equiv\xi$, $\vec{\xi_2}\equiv\vec{\eta}$, the associated Bel currents are expressed as

$$\vec{j}\left(R_{[2],[2]};\vec{\xi_L},\vec{\xi_M},\vec{\xi_N}\right)\qquad\left(=\vec{j}\left(R_{[2],[2]};\vec{\xi_{(L}},\vec{\xi_M},\vec{\xi_{N)}}\right)\right).\tag{4}$$

Of course, if our spacetime admits only a Killing vector field $\vec{\xi}$, then its associated Bel current is given by (4) with $L=M=N=1$ and $\vec{\xi_1}\equiv\vec{\xi}$.

Theorem 4.1 *Let* $\vec{\xi}$ *be a non-null Killing vector field. If* $\vec{\xi}$ *is integrable, then* $j_{[\beta}\left(R_{[2],[2]};\vec{\xi},\vec{\xi},\vec{\xi}\right)\xi_{\lambda]}=0,$

If the spacetime admits two independent non-null Killing vector fields $\vec{\xi}$ *and* $\vec{\eta}$*, let (4) be their associated Bel currents. If* $[\vec{\xi},\vec{\eta}]=0$ *and the group acts orthogonally transitively, then*

$$j_{[\beta}\left(R_{[2],[2]};\vec{\xi_L},\vec{\xi_M},\vec{\xi_N}\right)\xi_{\lambda}\eta_{\mu]}=0.\tag{5}$$

Equations (5) hold if and only if

$$j^{\alpha}\left(R_{[2],[2]};\vec{\xi_L},\vec{\xi_M},\vec{\xi_N}\right)=a_{LMN}(x^{\beta})\xi^{\alpha}+b_{LMN}(x^{\beta})\eta^{\alpha},\tag{6}$$

where the functions a's ($a_{(LMN)}=a_{LMN}$) and b's ($b_{(LMN)}=b_{LMN}$) must satisfy some relations involving the norms and the product of the Killing vectors to account for the symmetric character of the Bel tensor. But more importantly, taking the Lie derivative with respect to both Killing vector fields of equation (6), we have that

$$\mathcal{L}_{\vec{\xi}_P}\left(B^{\alpha}{}_{(\beta\lambda\mu)}\xi^{\beta}_{L}\xi^{\lambda}_{M}\xi^{\mu}_{N}\right)=\xi^{\rho}_{P}\nabla_{\rho}\left(a_{LMN}\right)\xi^{\alpha}+\xi^{\rho}_{P}\nabla_{\rho}\left(b_{LMN}\right)\eta^{\alpha},$$

which, since the Bel tensor is invariant under the action of isometries by construction and the group is Abelian and thus the left hand side of the equation vanishes, leads then to

$$\xi^{\rho}_{P}\nabla_{\rho}\left(a_{LMN}\right)=\xi^{\rho}_{P}\nabla_{\rho}\left(b_{LMN}\right)=0$$

for $P = 1, 2$. Taking now the divergence of equation (6) we have thus

$$\nabla_\rho j^\rho \left(R_{[2],[2]}; \vec{\xi_L}, \vec{\xi_M}, \vec{\xi_N} \right) = \xi^\rho_P \nabla_\rho \left(a_{LMN} \right) + \xi^\rho_P \nabla_\rho \left(b_{LMN} \right) = 0.$$

The cases when $\vec{\xi}$ and/or $\vec{\eta}$ is integrable can be treated a special cases of equation (6) with $LMN = 111$ with $b_{111} = 0$ and/or $LMN = 222$ with $a_{222} = 0$ respectively. All these results can be summarized as follows:

Corollary 4.1.1 *Let $\vec{\xi}$ be a non-null Killing vector field. If $\vec{\xi}$ is integrable then its corresponding Bel current $\vec{j}\left(R_{[2],[2]}; \vec{\xi}, \vec{\xi}, \vec{\xi}\right)$ is divergence-free.*

If the spacetime admits two independent non-null Killing vector fields $\vec{\xi}$ and $\vec{\eta}$, let (4) be their associated Bel currents. If $[\vec{\xi}, \vec{\eta}] = 0$ and the group acts orthogonally transitively, then the four coresponding Bel currents (4) are divergence-free.

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